APPROXIMATING JONES COEFFICIENTS AND OTHER LINK INVARIANTS BY VASSILIEV INVARIANTS

ILYA KOFMAN
Department of Mathematics, University of Maryland,
College Park, MD 20742, U.S.A.

YONGWU RONG
Department of Mathematics, George Washington University
Washington, DC 20052, U.S.A.

ABSTRACT

We find approximations by Vassiliev invariants for the coefficients of the Jones polynomial and all specializations of the HOMFLY and Kauffman polynomials. Consequently, we obtain approximations of some other link invariants arising from the homology of branched covers of links.

Keywords: Vassiliev invariant, Vandermonde matrix, branched cyclic cover

1. Introduction

A well-known conjecture in the theory of Vassiliev invariants is that these invariants are dense in the space of all numerical knot invariants. This was posed as a problem in [1] as follows: given any numerical knot invariant $\phi : L \to \mathbf{Q}$, does there exist a sequence of Vassiliev invariants $\{v_i : L \to \mathbf{Q}, i = 2, 3, 4, \ldots\}$ such that

$$\lim_{i\to\infty} v_i(L) = \phi(L)$$

In this note, we find approximations by Vassiliev invariants for the coefficients of the Jones polynomial and all specializations of the HOMFLY and Kauffman polynomials. Consequently, we obtain approximations of some other link invariants. This note is organized as follows: In Section 2, we show that every Jones coefficient is the limit of a sequence of Vassiliev invariants. In Section 3, given any d, for any Jones polynomial of degree bounded by d, we find an explicit finite formula for its coefficients in terms of Vassiliev invariants. In Section 4, we find an explicit infinite approximation for any Jones coefficient. In Section 5, we extend the results to any specialization of the HOMFLY and Kauffman polynomials: we find the finite formula for polynomials of bounded degree and the infinite formula for all polynomials. In Section 6, we find approximations by Vassiliev invariants for some

link invariants arising from the homology of branched covers of links. In Section 7, we discuss some conjectures related to approximations by Vassiliev invariants.

2. Approximating Jones coefficients by Vassiliev invariants: Existence theorem

Let $J_L(t)$ denote the Jones polynomial of a knot L. Suppose $J_L(t) = a_{-m}t^{-m} + \cdots + a_0 + \cdots + a_n t^n$, where a_{-m} and a_n are nonzero. We call the *degree* of the Laurent polynomial $d = \max(m, n)$. In [1], it was shown that if we let $t = e^x$,

$$J_L(e^x) = \sum_{i=0}^{\infty} \left(\frac{1}{i!} \sum_{k=-m}^{n} k^i a_k(L) \right) x^i = \sum_{i=0}^{\infty} v_i(L) x^i \tag{2.1}$$

then v_i is a Vassiliev invariant of order i. Henceforth, we will refer to $\{v_i\}$ as the Vassiliev invariants obtained from the coefficients of the expansion above.

This can be reformulated in terms of the following infinite matrix:

$$\begin{pmatrix} \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\ \cdots & (-3)^2 & (-2)^2 & (-1)^2 & 0 & 1^2 & 2^2 & 3^2 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ 2!v_2 \\ 3!v_3 \\ \vdots \end{pmatrix} \tag{2.2}$$

Recall that a Vandermonde matrix has the following form (see, e.g., [2]):

$$V(x_1, \ldots, x_r) = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_r \\ x_1^2 & \cdots & x_r^2 \\ \vdots & & \vdots \\ x_1^{r-1} & \cdots & x_r^{r-1} \end{pmatrix}$$

$$\det V(x_1, \ldots, x_r) = \prod_{i<j} (x_j - x_i)$$

Thus, $V(x_1, \ldots, x_r)$ is invertible if $x_i \neq x_j$ for all $i \neq j$. The matrix in (2.2) is a Vandermonde matrix for every finite square block which contains the first row. From the resulting system of linear equations, we obtain the following existence theorem:

Theorem 2.1. Given any knot L, let $J_L(t)$ be the Jones polynomial, and a_i be its i^{th} coefficient. Then for each i, $a_i(L)$ is the limit of a sequence of Vassiliev invariants.

Proof. For any coefficient a_i, we will define a sequence of Vassiliev invariants $\alpha_{1,i}, \alpha_{2,i}, \ldots$ and show that $\lim_{n\to\infty} \alpha_{n,i}(L) = a_i(L)$.

We now let $t = e^x$ and consider the expansion (2.1). If $n \geq d$, we obtain the following system of linear equations:

$$\left\{ \begin{array}{rlcl} a_{-n} & + \ldots + a_{-1} + a_0 + a_1 + \ldots + & a_n & = & v_0 \\ (-n)a_{-n} & + \ldots + (-1)a_{-1} + a_1 + \ldots + & na_n & = & v_1 \\ \vdots & & & & \vdots \\ (-n)^k a_{-n} & + \ldots + (-1)^k a_{-1} + a_1 + \ldots + & n^k a_n & = & v_k \end{array} \right. \tag{2.3}$$

with $2n+1$ variables $a_{-n}, \ldots, a_n$, and $k+1$ equations. When $k = 2n$, the system of linear equations has a unique solution since the coefficient matrix is an invertible finite block of the Vandermonde matrix from (2.2). Denote the solution by the vector $(\alpha_{n,-n}, \ldots, \alpha_{n,-1}, \alpha_{n,0}, \alpha_{n,1}, \ldots, \alpha_{n,n})$.

We claim $\lim_{n\to\infty} \alpha_{n,i}(L) = a_i(L)$. The claim follows immediately from the following lemma:

Lemma 2.1. For any knot L, $\alpha_{n,i}(L) = a_i(L)$ for all $n \geq d$.

When $n \geq d$, $J_{L,n} = J_L$. We let $\alpha_n(L) = (\alpha_{n,-n}(L), \ldots, \alpha_{n,0}(L), \ldots, \alpha_{n,n}(L))$ be given by the coefficients of $J_L(t)$. Therefore, $\alpha_n(L)$ is the unique solution satisfying the above system of linear equations. If we now fix i, for all $n \geq d$, it follows that $\alpha_{n,i}(L) = a_i(L)$. This completes the proof of the theorem. $\square$

Corollary 2.1. For any knot L and any fixed complex number z, $J_L(z)$ is a limit of Vassiliev invariants.

Proof. For each n, let

$$\alpha_n(L) = (\ldots, 0, \alpha_{n,-n}(L), \ldots, \alpha_{n,-1}(L), \alpha_{n,0}(L), \alpha_{n,1}(L), \ldots, \alpha_{n,n}(L), 0, \ldots)$$

be the sequence of infinite vectors as defined in Theorem 2.1. Let

$$g_n(L) = \alpha_{n,-n}(L)z^{-n} + \ldots + \alpha_{n,0}(L) + \ldots + \alpha_{n,n}(L)z^n.$$

Then g_n is a Vassiliev invariant since it is a linear combination of such invariants. By Lemma 2.1, when $n \geq d$, we have $\alpha_{n,i}(L) = a_i(L)$ for all i. Thus $g_n(L) = J_L(z)$. $\square$

3. Approximating Jones coefficients by Vassiliev invariants: Bounded degree case

For any given d (in particular, for any given knot), we can obtain explicit solutions to the linear system (2.3) and obtain a formula for all $\alpha_{n,i}$. We will use (2.2) and compute the inverse of the $(2d+1) \times (2d+1)$ Vandermonde matrix which is symmetric about the column with zeros. For this, we need the following generating function:

Definition 3.1.

$$f_{d,n}(v) = \prod_{\substack{j\neq n\\ j=-d}}^{d} \frac{v-j}{n-j} = \frac{(-1)^{n+d}}{(d+n)!\,(d-n)!} \prod_{\substack{j\neq n\\ j=-d}}^{d} (v-j)$$

The proof of the following proposition is immediate from the definition:

Proposition 3.1. For any $m \in \mathbf{Z}$ such that $-d \leq m \leq d$,

$$f_{d,n}(m) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Theorem 3.2. For any Jones polynomial of a knot of degree $\leq d$,

$$a_n = \sum_{i=0}^{2d} f_{d,n}^{(i)}(0) v_i, \quad \text{where } f_{d,n}(v) = \prod_{\substack{j\neq n\\ j=-d}}^{d} \frac{v-j}{n-j}$$

In other words, $\alpha_{d,n} = \sum_{i=0}^{2d} f_{d,n}^{(i)}(0) v_i$.

Proof. Let $c_{n,j} = \frac{1}{j!} f_{d,n}^{(j)}(0)$, the j^{th} coefficient of the polynomial $f_{d,n}(v)$.

$$\begin{pmatrix} c_{-d,0} & \cdots & c_{-d,2d} \\ \vdots & & \vdots \\ c_{d,0} & \cdots & c_{d,2d} \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 \\ -d & \cdots & 0 & \cdots & d \\ \vdots & & \vdots & & \vdots \\ (-d)^{2d} & \cdots & 0 & \cdots & d^{2d} \end{pmatrix} =$$

$$\begin{pmatrix} f_{d,-d}(-d) & \cdots & f_{d,-d}(d) \\ \vdots & & \vdots \\ f_{d,d}(-d) & \cdots & f_{d,d}(d) \end{pmatrix} = I_{(2d+1)\times(2d+1)}$$

Therefore,

$$\begin{pmatrix} a_{-d} \\ \vdots \\ a_0 \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} c_{-d,0} & \cdots & c_{-d,2d} \\ \vdots & & \vdots \\ c_{d,0} & \cdots & c_{d,2d} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ 2!v_2 \\ \vdots \\ (2d)!v_{2d} \end{pmatrix}$$

$\square$

With some extra notation, we can state the theorem more succinctly. Let $\mathcal{V}_i$ be the vector space of Vassiliev invariants spanned by v_i from (2.1). Consider the underlying vector space of the polynomial algebra $\mathbf{Q}[v]$ with basis $\{v^0, v^1, v^2, \ldots\}$. Let E be the vector space isomorphism $E : \mathbf{Q}[v] \to \oplus \mathcal{V}_i$, where $E(v^i) = i!v_i$. We therefore obtain:

$$a_n = E(f_{d,n}(v))$$

Example 3.1. As in Section 2, let $\alpha_{d,n}$ denote the n^{th} coefficient of the Jones polynomial of degree $\leq d$.

$$
\begin{aligned}
\alpha_{2,0} &= \tfrac{1}{2!2!}(4-5v_2+v_4) = \tfrac{1}{2!2!}E\left((v-2)(v-1)(v+1)(v+2)\right) \\
\alpha_{3,0} &= \tfrac{1}{3!3!}(36-49v_2+14v_4-v_6) = \\
&= -\tfrac{1}{3!3!}E\left((v-3)(v-2)(v-1)(v+1)(v+2)(v+3)\right) \\
\alpha_{2,1} &= \tfrac{1}{1!3!}(4v_1+4v_2-v_3-v_4) = -\tfrac{1}{1!3!}E\left((v-2)(v)(v+1)(v+2)\right) \\
\alpha_{3,1} &= \tfrac{1}{2!4!}(36v_1+36v_2-13v_3-13v_4+v_5+v_6) = \\
&= \tfrac{1}{2!4!}E\left((v-3)(v-2)(v)(v+1)(v+2)(v+3)\right)
\end{aligned}
$$

Remark 3.1. From Theorem 3.2, we obtain a formula for any $v_i(L)$ in terms of $v_0(L), \ldots, v_{2d}(L)$. Namely,

$$v_i(L) = \frac{1}{i!}\sum_{k=-m}^{n} k^i a_k(L) = \sum_{j=0}^{2d}\left(\frac{1}{i!}\sum_{k=-m}^{n} k^i f_{d,k}^{(j)}(0)\right) v_j(L)$$

Similarly, in [3] it was shown, without an explicit formula, that for any knot L, there exists N, such that all $v_i(L)$ are determined by $v_0(L), \ldots, v_N(L)$.

4. Approximating Jones coefficients by Vassiliev invariants: Infinite case

In this section, we formally let $d \to \infty$ to find the correct formula for the coefficients of an arbitrary Jones polynomial of a knot, and then prove that the resulting series of Vassiliev invariants converges. We also extend the vector space isomorphism $E : \mathbf{Q}[[v]] \to \prod \mathcal{V}_i$, where $E(v^i) = i!v_i$.

We first consider a_0:

$$f_{d,0}(v) = \prod_{\substack{j\neq 0\\ j=-d}}^{d} \frac{v-j}{-j} = \prod_{\substack{j\neq 0\\ j=-d}}^{d}\left(1-\frac{v}{j}\right) = \prod_{j=1}^{d}\left(1-\frac{v}{j}\right)\left(1+\frac{v}{j}\right) = \prod_{j=1}^{d}\left(1-\frac{v^2}{j^2}\right)$$

We recall the Weierstrass product factorization for entire functions:

$$\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right) = \frac{\sin \pi z}{\pi z}$$

We define $f_{\infty,0}(v) = \lim_{d\to\infty} f_{d,0}(v)$, so we obtain

$$f_{\infty,0}(v) = \frac{\sin \pi v}{\pi v} = 1 - \frac{(\pi v)^2}{3!} + \frac{(\pi v)^4}{5!} - \frac{(\pi v)^6}{7!} + \cdots$$

Formally (we prove convergence below), we obtain the following beautiful formula:

$$a_0 = E(f_{\infty,0}(v)) = E(\frac{\sin \pi v}{\pi v}) = v_0 - \frac{\pi^2}{3} v_2 + \frac{\pi^4}{5} v_4 - \frac{\pi^6}{7} v_6 + \cdots \tag{4.4}$$

We now consider a_n:

$$f_{d,n}(v) = \prod_{\substack{j \neq n \\ j=-d}}^{d} \frac{v-j}{n-j} = \prod_{j=-d}^{n-1} \frac{v-j}{n-j} \prod_{j=n+1}^{d} \frac{v-j}{n-j}$$

Let $k = n - j$ in the first product, and $k = j - n$ in the second product, so we obtain

$$f_{d,n}(v) = \prod_{k=1}^{d+n} \frac{v+k-n}{k} \prod_{k=1}^{d-n} \frac{v-k-n}{-k} = \prod_{k=1}^{d-n} (1 - \frac{(v-n)^2}{k^2}) \prod_{k=d-n+1}^{d+n} (1 + \frac{v-n}{k})$$

The second product is finite: let $l = k - d$, then we obtain $\prod_{l=-n+1}^{n} (1 + \frac{v-n}{l+d})$. For any n, as $d \to \infty$ we can easily see that this product converges to 1. The first product converges for all n by the same argument as above. This suggests the following theorem:

Theorem 4.3.

$$a_n = E(f_{\infty,n}(v)) = \sum_{i=0}^{\infty} f_{\infty,n}^{(i)}(0) v_i, \text{ where } f_{\infty,n}(v) = \begin{cases} \frac{\sin \pi(v-n)}{\pi(v-n)} & \text{if } v \neq n \\ 1 & \text{if } v = n \end{cases}$$

Proof. For any given knot L, the Jones polynomial has finite degree d.

$$\sum_{i=0}^{\infty} f_{\infty,n}^{(i)}(0) v_i(L) = \sum_{i=0}^{\infty} f_{\infty,n}^{(i)}(0) \left(\frac{1}{i!} \sum_{k=-d}^{d} k^i a_k(L) \right)$$

$$= \sum_{k=-d}^{d} a_k(L) \left(\sum_{i=0}^{\infty} \frac{1}{i!} f_{\infty,n}^{(i)}(0) k^i \right)$$

$$= \sum_{k=-d}^{d} a_k(L) f_{\infty,n}(k) = a_n(L)$$

□

Example 4.2.

$$a_1 = v_1 + 2v_2 + (3! - \pi^2) v_3 + (4! - 4\pi^2) v_4 + (5! + \pi^4 - 20\pi^2) v_5 + \cdots$$

Corollary 4.2. Any Jones coefficient of a knot can be approximated by Vassiliev invariants $\tilde{v}_i$ of order i:

$$a_n = \lim_{i \to \infty} \tilde{v}_i, \quad \text{where } \tilde{v}_i = \sum_{j=0}^{i} f_{\infty,n}^{(j)}(0) v_j$$

Remark 4.2. Jones coefficients can be shown not to be Vassiliev invariants by considering twist sequences [4]. Let T_m be the $(2, 2m+1)$-torus knot. By Theorem 2.2.1 of [4], the restriction of any Vassiliev invariant to the sequence $\{T_m\}$ is a polynomial in m. However, $J_{T_m}(t) = -t^m(t^{2m+1} - \ldots + t^3 - t^2 - 1)$. Thus for $n \geq 0$, $a_n(T_m) = 0$ for $m < \frac{n-1}{3}$ or $m > n$. If $a_n(T_m)$ were a polynomial in m, it would be zero on T_m, which is clearly false. Similarly, we can take mirror images of T_m to show that for all $n < 0$, a_n is not a Vassiliev invariant. (See also [5].)

Trapp [4] also showed that if a sequence of Vassiliev invariants converges uniformly for all knots, then the limit is also of finite type. Because the Jones coefficients are not of finite type, the pointwise limits above cannot be uniformly convergent for all knots. Indeed, the proof of Theorem 4.3 requires us to first choose a particular knot.

Remark 4.3. The function $f_{\infty,n}(v)$ is not unique, because the infinite Vandermonde matrix can have infinitely many left inverses. Since every Jones polynomial has finite degree d, but Vassiliev invariants may be nonzero for arbitrary orders, we can view the infinite matrix as a linear operator $\oplus_{i=-\infty}^{\infty}\mathbf{Q} \to \prod_{i=0}^{\infty}\mathbf{Q}$, so it can have infinitely many left inverses, but no right inverse. The proof of Theorem 4.3 only requires that $f_{\infty,n}(v)$ has a Taylor expansion about zero, and that $f_{\infty,n}(m) = \delta_{m,n}$, the Kronecker pairing for all $m, n \in \mathbf{Z}$. If we also insist that $f_{\infty,n}(v) = \lim_{d\to\infty} f_{d,n}(v)$, then the generating function depends on how we select invertible finite blocks to exhaust the infinite Vandermonde matrix.

Remark 4.4. Given an infinite sequence $\{v_n(L)\}$, we can also approximate the degree d of the Jones polynomial by functions of finite type invariants:

$$d = \lim_{n\to\infty} \sqrt[n]{\left|\sum_{k=-d}^{d} k^n a_k(L)\right|} = \lim_{n\to\infty} \sqrt[n]{n!|v_n|}$$

Remark 4.5. All of the results above carry over with slight modifications to links. The Jones polynomial of a link may be a Laurent polynomial times $\sqrt{t}$, so instead of (2.2), the matrix and resulting formulas appear as a special case of Theorem 5.4.

5. Approximations of coefficients of specializations of HOMFLY and Kauffman polynomials

The Jones polynomial is a specialization of both the HOMFLY and Kauffman two-variable polynomials, $H_L(a,z)$ and $F_L(a,z) \in \mathbf{Z}[a^{\pm1}, z^{\pm1}]$. We consider an infinite sequence of one-variable specializations of the HOMFLY polynomial. The same result and proof applies to the Kauffman polynomial as well. Let $N \in \mathbf{Z}\backslash\{0\}$.

$$a = t^{N/2}, z = t^{1/2} - t^{-1/2} \quad \Rightarrow \quad H_L^N(t) \in Z[t^{\pm\frac{1}{2}}] \tag{5.5}$$

The Jones polynomial is obtained at $N=-2$. Now, suppose $H_L^N(t) = b_{-m}^N t^{-m/2} + \cdots + b_0^N + \cdots + b_n^N t^{n/2}$. Let $d = \max(m,n)$. In [1], it was shown that if we let $t = e^x$,

$$H_L^N(e^x) = \sum_{i=0}^{\infty} \left(\frac{1}{i!} \sum_{k=-m}^{n} \left(\frac{k}{2}\right)^i b_k^N(L) \right) x^i = \sum_{i=0}^{\infty} v_i^N(L) x^i \tag{5.6}$$

then v_i^N is a Vassiliev invariant of order i.

Theorem 5.4. For any link L and any $N \in \mathbf{Z}\backslash\{0\}$, let its N^{th} HOMFLY polynomial be $H_L^N(t) = \sum_{n=-d}^{d} b_n^N t^{n/2}$. Let $E^N(v^i) = i! v_i^N$. Then,

$$b_n^N = E^N(f_{d,n}(v)) = \sum_{i=0}^{2d} f_{d,n}^{(i)}(0) v_i^N, \quad \text{where } f_{d,n}(v) = \prod_{\substack{j \neq n \\ j=-d}}^{d} \frac{2v-j}{n-j}$$

$$b_n^N = E^N(f_{\infty,n}(v)) = \sum_{i=0}^{\infty} f_{\infty,n}^{(i)}(0) v_i^N, \text{ where } f_{\infty,n}(v) = \begin{cases} \frac{\sin \pi(2v-n)}{\pi(2v-n)} & \text{if } 2v \neq n \\ 1 & \text{if } 2v = n \end{cases}$$

Proof. The proof is just a modification of the proof for the Jones polynomial. Instead of (2.2), we have

$$\begin{pmatrix} \cdots & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & -1 & -1/2 & 0 & 1/2 & 1 & \cdots \\ \cdots & (-1)^2 & (-1/2)^2 & 0 & (1/2)^2 & 1^2 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} \vdots \\ b_{-1}^N \\ b_0^N \\ b_1^N \\ \vdots \end{pmatrix} = \begin{pmatrix} v_0^N \\ v_1^N \\ 2! v_2^N \\ 3! v_3^N \\ \vdots \end{pmatrix}$$

To find the inverse of both the infinite matrix and any invertible finite block, we just need the following:

Proposition 5.2. For any $m \in \mathbf{Z}$ such that $-d \leq m \leq d$,

$$f_{d,n}\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

and similarly for $f_{\infty,n}(\frac{m}{2})$ for any $m \in \mathbf{Z}$. $\square$

Corollary 5.3. For any link L, any fixed complex number z, and any $N \in \mathbf{Z}\backslash\{0\}$, $H_L^N(z)$ is a limit of Vassiliev invariants.

Proof. The proof is the same as the proof of Corollary 2.1. $\square$

Remark 5.6. Following [1], let $F_L(a,z)$ denote the Dubrovnik version of the Kauffman polynomial. To obtain one-variable specializations, for $N \in \mathbf{Z}\backslash\{0\}$, set $a = t^N, z = t - t^{-1}$ and let $F_L^N(t)$ denote the N^{th} Kauffman polynomial. Theorem 5.4 and Corollary 5.3 carry over to $F_L^N(t)$.

6. Approximations of other link invariants

Let $\Sigma_n(L)$ denote the n-fold branched cover of a link L. Let $Q_L(x)$ be the specialization at $a = 1$ of the standard Kauffman polynomial. This link polynomial satisfies the skein relation $Q_{L_+} + Q_{L_-} = x(Q_{L_0} + Q_{L_\infty})$ and $Q(\text{unknot}) = 1$ [6, 7]. A lot of information about $H_1(\Sigma_n(L))$ can be obtained by evaluating link polynomials of L at special values. In fact, it seems reasonable to conjecture that $H_1(\Sigma_2(L), \mathbf{Z})$ is determined by Q_L [8]. We summarize the results below. All the evaluations can be found in [9], except for $Q_L(2\cos\frac{2\pi}{5})$ which is given in [10] and [8].

For the Jones polynomial, the most interesting values to evaluate are $t = e^{\frac{2\pi i}{r}}$, where r is a positive integer.

Table 1. Jones polynomial at $t = e^{\frac{2\pi i}{r}}$

r	$J_L(e^{\frac{2\pi i}{r}})$
$r=1$	$(-2)^{\ell-1}$
$r=2$	$\lvert H_1(\Sigma_2, \mathbf{Z})\rvert = \det L$
$r=3$	1
$r=4$	$(-\sqrt{2})^{(\ell-1)}(-1)^{\text{Arf}(L)}$ if $\text{Arf}(L)$ exists, 0 if $\text{Arf}(L)$ undefined
$r=6$	$(-\sqrt{3})^{\dim(H_1(\Sigma_2,\mathbf{Z}_3))}(-i)^w$

In the table, ℓ is the number of components of L, $\text{Arf}(L)$ is the Arf invariant of L, and $w \in \mathbf{Z}_4$ is the Witt class of the Seifert form mod 3 of L.

Because the Jones polynomial is a specialization of the HOMFLY polynomial, the table also gives evaluations of the HOMFLY polynomial. Another interesting value not listed above is $H_L(-i, i) = (i\sqrt{2})^{\dim(H_1(\Sigma_3,\mathbf{Z}_2))}$. Let us also recall that $|H_1(\Sigma_n, \mathbf{Z})| = |\prod \Delta_L(r_i)|$, where r_i's are the nth roots of unity.

For the Q-polynomial, the interesting values are at $x = 2\cos\frac{2\pi}{r} = q + q^{-1}$, where $q = e^{\frac{2\pi i}{r}}$.

Table 2. Q-polynomial at $x = 2\cos\frac{2\pi}{r}$

r	$Q_L(2\cos\frac{2\pi}{r})$
$r=1$	$(-1)^{\ell-1}\lvert\det L\rvert^2$
$r=2$	$(-2)^{\ell-1}$
$r=3$	$(-3)^{\dim(H_1(\Sigma_2,\mathbf{Z}_3))}$
$r=4$	undefined
$r=5$	$(-1)^{t_5(L)}(\sqrt{5})^{\dim(H_1(\Sigma_2,\mathbf{Z}_5))}$
$r=6$	1

Here t_5 is 0 or 1, and can be written in terms of the Seifert form mod 5 [8].

Theorem 6.5. Let ϕ be any of the following knot invariants: $\det L, |H_1(\Sigma_2, \mathbf{Z}_p)|$ where $p = 3, 5$, $|H_1(\Sigma_3, \mathbf{Z}_2)|$, $|H_1(\Sigma_n, \mathbf{Z})|$. Then
(a) ϕ is not a Vassiliev invariant.
(b) ϕ is a limit of functions of Vassiliev invariants.

Proof. (a) As remarked in [11], a knot invariant ϕ is not a Vassiliev invariant if there is a knot K with $\phi(K\#K') \neq \phi(\text{unknot})$ for all knots K'. Now let ϕ be, for example, the order of $H_1(\Sigma_2, \mathbf{Z}_p)$. Let K be a knot with $\phi(K) = p$ (e.g., any 2-bridge knot for which $\Sigma_2(K)$ is the lens space $L_{p,q}$). Then $\phi(K\#K') = \phi(K)\phi(K') \neq \phi(\text{unknot})$. Similarly, $|H_1(\Sigma_3, \mathbf{Z}_2)|$ and $|H_1(\Sigma_n, \mathbf{Z})|$ are not Vassiliev invariants.

(b) The following equations come from the tables and comments above. Together with Corollary 2.1 and Corollary 5.3, they imply that ϕ is a limit of functions of Vassiliev invariants.

$$\begin{array}{lcl}
|H_1(\Sigma_2, \mathbf{Z})| & = & J_L(-1) \\
|H_1(\Sigma_2, \mathbf{Z}_3)| & = & |J_L(e^{\frac{\pi i}{3}})|^2 \\
|H_1(\Sigma_2, \mathbf{Z}_5)| & = & |Q_L(2\cos(2\pi/5))|^2 \\
|H_1(\Sigma_3, \mathbf{Z}_2)| & = & |H_L(-i, i)|^2 \\
|H_1(\Sigma_n, \mathbf{Z})| & = & |\prod \Delta_L(r_i)|, \text{ where } r_i\text{'s are the } n\text{th roots of unity.}
\end{array}$$

Note from (5.5), we obtain that $H_L(-i, i)$ is $H_L^N(t)$, where $N = 9,\ t = e^{i\pi/3}$.

As in Section 5, let $F_L^N(t)$ denote the N^{th} Dubrovnik Kauffman polynomial. By a change of variables, $Q_L(x) = (-1)^\ell F_L(i, -ix)$ [12]. Thus, by Remark 5.6 we obtain that $Q_L(2\cos(2\pi/5))$ is $(-1)^\ell F_L^N(t)$, where $N = -5,\ t = -ie^{2i\pi/5}$.

Since the coefficients of the Alexander-Conway polynomial are Vassiliev invariants [13], by an argument similar to Corollary 2.1 we obtain that $\prod \Delta_L(r_i)$ is a limit of Vassiliev invariants. Thus, $|\prod \Delta_L(r_i)|$ is a limit of the absolute value function of Vassiliev invariants. $\square$

Remark 6.7. For all ϕ except for $\phi = |H_1(\Sigma_n, \mathbf{Z})|$, we can show that ϕ is actually a limit of Vassiliev invariants. For example, in the case of $|H_1(\Sigma_2, \mathbf{Z}_3)|$, it follows from Corollary 2.1 and the equation: $|H_1(\Sigma_2, \mathbf{Z}_3)| = J_L(e^{\frac{\pi i}{3}})J_L(e^{-\frac{\pi i}{3}})$, since the two factors on the right are complex conjugates.

For other functions, e.g., $\phi = \dim(H_1(\Sigma_2, \mathbf{Z}_3))$, the argument above can only show that ϕ is a limit of functions of Vassiliev invariants. Note that in the case of $|H_1(\Sigma_n, \mathbf{Z})|$, the n-th roots of unity all appear in conjugate pairs, thus the product $\prod \Delta_L(r_i)$ can be arranged in conjugate pairs. It follows that the absolute value sign is not needed, except for $\Delta(-1)\Delta(1)$.

Remark 6.8. Another knot invariant which is not a Vassiliev invariant, but is a limit of Vassiliev invariants is $tri(L)$, the number of 3-colorings of L. This follows from a result of Przytycki [14, 15]: $tri(L) = 3|J_L(e^{\frac{\pi i}{3}})|^2$.

7. Conclusion

Here, we make some final remarks on approximations by Vassiliev invariants. Our work is motivated by the following two equivalent conjectures (see [16]):

Conjecture 7.1. Vassiliev invariants separate knots. That is, for any two knots K_1 and K_2, there is a Vassiliev invariant v with $v(K_1) \neq v(K_2)$.

Conjecture 7.2. Every knot invariant is a limit of Vassiliev invariants. That is, for any knot invariant ϕ, there is a sequence of Vassiliev invariants v_n with $\phi(L) = \lim_{n\to\infty} v_n(L)$ for all knots L.

We have verified that a number of knot invariants (e.g., coefficients of link polynomials) are indeed limits of Vassiliev invariants. Some other knot invariants (e.g., the degree of the Jones polynomial) are proved to be limits of functions of Vassiliev invariants. In light of this, we propose:

Conjecture 7.3. Every knot invariant is a limit of functions of Vassiliev invariants. That is, for any knot invariant ϕ, there is a sequence of functions f_n and Vassiliev invariants v_n with $\phi(L) = \lim_{n\to\infty} f_n(v_n)(L)$ for all knots L.

In general, if f is an analytic function, v is a Vassiliev invariant, then it is not hard to show that $f(v)$ is a limit of Vassiliev invariants. Consequently, a limit of analytic functions of Vassiliev invariants is in fact a limit of Vassiliev invariants. However, this is not clear if the functions are not analytic functions. This is the case, for example, for the degree of the Jones polynomial, where the functions f_n are the nth root function.

One good aspect of Conjecture 7.3 is that it is easier to verify than Conjecture 7.2 for a given knot invariant, but is still strong enough to imply Conjecture 7.1. Therefore by [16],

$$\text{Conjecture 7.2} \Rightarrow \text{Conjecture 7.3} \Rightarrow \text{Conjecture 7.1} \Rightarrow \text{Conjecture 7.2}.$$

Acknowledgements

We would like to thank Dror Bar-Natan, Xiao-Song Lin, Ted Stanford, and Ed Swartz for helpful discussions. The first author was partially supported by NSF grant DMS-98-03518. The second author was partially supported by NSF grant DMS-97-29992 while visiting the Institute for Advanced Study.

References

[1] J. Birman and X.-S. Lin, *Knot polynomials and Vassiliev's invariants,* Invent. Math. **111** (1993) 225-270.

[2] M. Fiedler, **Special matrices and their applications in numerical mathematics,** Martinus Nijhoff Publishers (1986), p.159.

[3] L. Kauffman, M. Saito, S. Sawin, *On finiteness of certain Vassiliev invariants,* J. Knot Theory and its Ramifications **6** (1997), no. 2, 291-297.

[4] R. Trapp, *Twist sequences and Vassiliev invariants,* J. Knot Theory and its Ramifications **3** (1994), no. 3, 391-405.

[5] J. Zhu, *On Jones knot invariants and Vassiliev invariants,* New Zealand J. Math. **27** (1998), no. 2, 293–299.
[6] R. Brandt, W. Lickorish and K. Millett, *A polynomial invariant for unoriented knots and links,* Invent. Math. **84** (1986) 563-573.
[7] C. F. Ho, *A new polynomial invariant for knots and links,* Preliminary report, Amer. Math. Soc. Abstracts **6** (1985), 300.
[8] Y. Rong, *The Kauffman polynomial and the two-fold cover of a link,* Indiana Univ. Math. J. **40** (1991), no. 1, 321-331.
[9] W. Lickorish, *Polynomials for links,* Bull. London Math. Soc. **20** (1988) 558-588.
[10] V. Jones, *On a certain value of the Kauffman polynomial,* Comm. Math. Phys. **125** (1989) 459-467.
[11] T. Stanford and R. Trapp, *On knot invariants which are not of finite type,* Preprint (1999).
[12] L. Kauffman, **Knots and Physics,** World Scientific (1993), p.216.
[13] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995), no. 2, 423-472.
[14] J.Przytycki, *Elementary conjectures in classical knot theory,* in **Quantum Topology,** ed. L. Kauffman, R. Baadhio, Series on Knots and Everything, vol. 3, World Scientific (1993), pp.292-320.
[15] J.Przytycki, *3-coloring and other elementary invariants of knots,* in **Knot Theory,** ed. V. Jones, et. al., Banach Center Publ., vol. 42, Inst. of Math. Polish Academy of Sciences (1998), pp.275-295.
[16] P. Røgen, *On density of the Vassiliev invariants,* J. Knot Theory and its Ramifications **8** (1999), no. 2, 249-252.